Towards a Functor Between Affine and Finite Hecke Categories in Type A

by

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Abstract

In this thesis we construct a functor from the perfect subcategory of the coherent version of the affine Hecke category in type A to the finite constructible Hecke category, partly categorifying a certain natural homomorphism of the corresponding Hecke algebras. This homomorphism sends generators of the Bernstein's commutative subalgebra inside the affine Hecke algebra to Jucys-Murphy elements in the finite Hecke algebra. Construction employs the general strategy devised by Bezrukavnikov to prove the equivalence of coherent and constructible variants of the affine Hecke category. Namely, we identify an action of the category $\mathrm{Rep}(\mathrm{GL}_n)$ on the finite Hecke category, and lift this action to a functor from the perfect derived category of the Steinberg variety, by equipping it with various additional data.

Thesis Supervisor: Roman Bezrukavnikov
Title: Professor of Mathematics

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Dedicated to my grandmother, Elena.

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Chapter 1

Introduction

1.1 Algebraic setting

This thesis is concerned with the categorification of a certain homomorphism between the affine and finite Hecke algebras in type A, in the setting of known geometric categorifications of the algebras themselves.

The homomorphism in question may be most easily seen on the level of the corresponding braid groups. If we interpret the braid group B_n on n strands as the fundamental group of the configuration space $\mathrm{Conf}_n(\mathbb{C})$ of n points in the complex plane $\mathbb{C}$, and the affine braid group B_n^{aff} as the fundamental group of the configuration space $\mathrm{Conf}_n(\mathbb{C}^*)$ of n points in the punctured complex plane $\mathbb{C}^*$, based at some configuration ζ not meeting $0 \in \mathbb{C}$, then the inclusion $\mathbb{C}^* \to \mathbb{C}$ gives the homomorpism

$$B_n^{\mathrm{aff}} \simeq \pi_1(\mathrm{Conf}_n(\mathbb{C}^*), \zeta) \to \pi_1(\mathrm{Conf}_n(\mathbb{C}), \zeta) \simeq B_n,$$

see Figure 1-1. This descends to the homomorphism

$$\Pi : \tilde{\mathbb{H}}_n^{\mathrm{aff}} \to \mathbb{H}_n$$

between the corresponding extended affine Hecke algebra of the reductive group GL_n and the finite Hecke algebra. Recall that $\tilde{\mathbb{H}}_n^{\mathrm{aff}}$ contains $\mathbb{H}_n$ as a subalgebra. It also

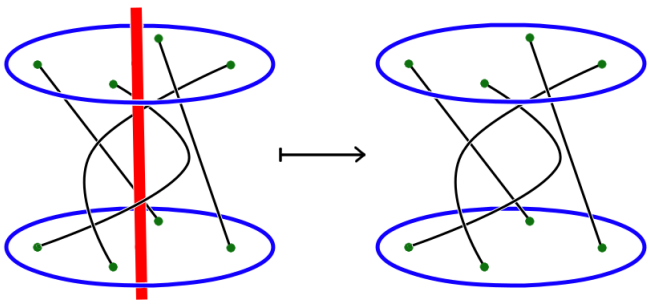

Figure 1-1: Homomorphism on the level of braid groups

contains a large commutative subalgebra, defined by Bernstein, which is identified with the group algebra of the (co-)weight lattice of GL_n. The homomorphism Π has the following remarkable property: under some choice $\{e_1, \dots, e_n\}$ of generators in this lattice, the corresponding Bernstein generators $\{\theta_1, \dots, \theta_n\}$ map to the multiplicative Jucys-Murphy elements $\{L_1, \dots, L_n\}$ in the finite Hecke algebra. Π is identity when restricted to $\mathbb{H}_n \subset \tilde{\mathbb{H}}_n^{\mathrm{aff}}$.

Symmetric polynomials in $\theta_i^{\pm 1}$ are known to span the center $\mathfrak{Z}(\tilde{\mathbb{H}}_n^{\mathrm{aff}})$ of $\tilde{\mathbb{H}}_n^{\mathrm{aff}}$, see [Lus83], and so are symmetric polynomials in L_i for $\mathbb{H}_n$, see [DJ87]. We get the following diagram

$$\begin{array}{ccc} \mathfrak{Z}(\tilde{\mathbb{H}}_n^{\mathrm{aff}}) & \hookrightarrow & \tilde{\mathbb{H}}_n^{\mathrm{aff}} \\ {\scriptstyle \Pi|_{\mathfrak{Z}}}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathfrak{Z}(\mathbb{H}_n) & \hookrightarrow & \mathbb{H}_n \end{array} \tag{1.1}$$

All corners of this diagram have their geometric categorifications.

1.2 Categorifications

In what follows we work over the field of the complex numbers $\mathbb{C}$. All the categories considered are $\mathbb{C}$-linear and all the varieties are defined over $\mathbb{C}$.

Categorification of the finite Hecke algebra comes in many forms, of which we will use some completed variant of the monodromic category $\hat{D}_f$ of constructible sheaves on the base affine space for GL_n. The categorification of the center is the completed variant of the derived category of unipotent character sheaves on GL_n, $D^b(\hat{\mathcal{CS}})$: in fact, it is known that the appropriate variant of the category of character sheaves is the Drinfeld center of the finite Hecke category. See [BFO12] for the statement in the setting of abelian categories of D-modules, [BZN09] for the statement in the derived setting, and [Lus15] for the related statement in arbitrary characteristic. The inclusion of the center is given by the so-called Harish-Chandra functor, which we denote $\mathfrak{hc}$, introduced by Lusztig. See also [Gin89], where the relation to central objects in the Hecke category is noted.

While categorifications of the finite Hecke category all come from the world of constructible sheaves (or D-modules), one can obtain a categorification of the extended affine Hecke algebra both in constructible and coherent settings. Namely, our constructible avatar of the affine Hecke category is the completed monodromic variant of the derived category of constructible sheaves on the affine flag variety of GL_n, equivariant with respect to the (radical of) Iwahori subgroup, which we will denote simply by $D_I^b(\mathrm{Fl}^{\mathrm{aff}})$ in this introduction. It is a result of [Bez16], that the constructible affine Hecke category is equivalent to (the completed variant of) the derived category of the equivariant coherent sheaves on the Steinberg variety, which we will denote by $D^b(Coh_G(\hat{\mathrm{St}}))$ in this introduction. Under this equivalence, the structure sheaf $\mathcal{O}_{\hat{\mathrm{St}}}$ corresponds to the pro-unipotent tilting sheaf of the open orbit, $\hat{\mathcal{T}}_{w_0}$.

Categorification of the center of the affine Hecke algebra is known as a Satake category $\mathcal{P}_{G(\mathcal{O})}(Gr)$ of equivariant perverse sheaves on the affine Grassmannian cor-

responding to GL_n, see [Lus83], [MV07]. It was shown in op.cit., that this category is equivalent to the category $\mathrm{Rep}(\mathrm{GL}_n)$ of representations of GL_n. The corresponding inclusion was categorified in [Gai01] (note that $\mathrm{Rep}(\mathrm{GL}_n)$ is much smaller than the actual Drinfeld center of the affine Hecke category). All the categories discussed are equipped with the monoidal structure, given by convolution, which we denote by $*$ in this introduction. We obtain the following diagram:

$$\begin{array}{ccccccc} \mathrm{Rep}(\mathrm{GL}_n) & \xrightarrow[\sim]{[\mathrm{MV07}]} & \mathcal{P}_{G(\mathcal{O})}(Gr) & \xrightarrow{[\mathrm{Gai01}]} & D^b_I(\mathrm{Fl}^{\mathrm{aff}}) & \xrightarrow[\sim]{[\mathrm{Bez16}]} & D^b(Coh_G(\hat{\mathrm{St}})) \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & D^b(\hat{\mathcal{CS}}) & \xrightarrow{\hbar\mathfrak{c}} & \hat{D}_f & & \end{array} \tag{1.2}$$

Conjecture. *There are monoidal functors replacing dashed arrows in the diagram (1.2), making it commutative and categorifying the diagram (1.1).*

In this thesis, we prove the following weakened form of this conjecture. We have a perfect variant $D^G_{\mathrm{perf}}(\hat{\mathrm{St}})$ of the coherent Hecke category. Since the Steinberg variety is singular, it is not equivalent to the full derived category. Our main result is the following

Theorem (see Theorems 1, 3)**.** *There are monoidal functors ϖ_Z, ϖ, making the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{Rep}(GL_n) & \xrightarrow{[Gai01](\cdot)*\mathcal{O}_{\hat{St}}} & D^G_{perf}(\hat{St}) \\ {\scriptstyle\varpi_Z}\downarrow & & \downarrow{\scriptstyle\varpi} \\ D^b(\hat{\mathcal{CS}}) & \xrightarrow{\hbar\mathfrak{c}(\cdot)*\hat{\mathcal{T}}_{w_0}} & \hat{D}_f \end{array} \tag{1.3}$$

It is a categorification of a part of the diagram (1.1).

In the rest of the introduction, we describe the plan of the proof of this theorem.

1.3 Jucys-Murphy sheaves

The fact that Π is defined already for braid groups allows us to identify the desired images of the so-called Wakimoto sheaves categorifying Bernstein's generators.

We have a standard basis $\{t_w\}$ of the finite Hecke algebra, indexed by elements w of the Weyl group. A choice of a set of simple reflections $\{s_i\}$ gives a set of generators $t_i := t_{s_i}$. One can pick the generators $\{\theta_i\}$ of Bernstein's commutative subalgebra so that

$$\Pi(\theta_1) = t_e = 1 =: L_1,$$

$$\Pi(\theta_{k+1}) = t_k L_k t_k =: L_{k+1}.$$

Elements L_k are well-known in the study of Hecke algebras, and are called the multiplicative Jucys-Murphy elements. See [IO05] for a review.

Basis $\{t_w\}$ has a natural categorical analogue in the Hecke category $\hat{D}_f$, given by the so-called standard sheaves Δ_w (for the precise definition and notations see the main body of the paper). We can readily define

$$\mathbb{L}_1 := \Delta_e,$$

$$\mathbb{L}_{k+1} := \Delta_{s_k} * \mathbb{L}_k * \Delta_{s_k}.$$

We call the complexes $\mathbb{L}_k$ Jucys-Murphy sheaves.

Remark. Let w_0 be the longest element of the Weyl group of GL_n. Convolution with the complex $\Delta_{w_0} * \Delta_{w_0}$ defines the inverse of the Serre functor $\mathcal{S}_n$ on the Hecke category, see [BBM04]. Let w_0' be the longest element of the Weyl group of $\mathrm{GL}_{n-1} \times \mathbb{G}_m$ embedded into GL_n as a block-diagonal subgroup. We have

$$\mathbb{L}_n * \mathcal{F} = \Delta_{w_0}^{*2} \Delta_{w_0'}^{*(-2)} * \mathcal{F} = \mathcal{S}_n^{-1} \mathcal{S}_{n-1}(\mathcal{F}),$$

so n^{th} Jucys-Murphy sheaf can be interpreted as a "ratio" of two Serre functors. Compare [BK90], Proposition 3.9. We don't make any use of this fact below.

1.4 Plan of the categorification of the central arrow

The main contribution of this thesis is Theorem 1, which may be of independent interest. In it, we construct a functor from $\mathrm{Rep}(\mathrm{GL}_n)$ to the completed derived category of unipotent character sheaves. We use the following universal property of the category $\mathrm{Rep}(\mathrm{GL}_n)$: to give a monoidal functor from $\mathrm{Rep}(\mathrm{GL}_n)$ to a symmetric monoidal pseudo-abelian category $\mathcal{C}$, it is enough to specify an object $\mathcal{V}$ of $\mathcal{C}$, satisfying $\wedge^{n+1}\mathcal{V} = 0$ and such that $\wedge^n \mathcal{V}$ is invertible. It will be an object corresponding to the standard representation V_n of $\mathrm{GL}_n = \mathrm{GL}(V_n)$, from which one gets other representations by applying Schur functors and tensoring with powers of $\wedge^n V_n$. See Section 2.4.

We will apply this reasoning to the derived category of adjoint-equivariant constructible sheaves on GL_n. This category is monoidal, product being given by convolution, and can be equipped with a symmetric braided structure, see Section 3.5.

In our algebraic picture, object $\mathcal{V}$ should correspond to the element

$$\Pi\left(\sum_i \theta_i\right) = \sum_i L_i.$$

We would expect, then, our object $\mathcal{V}$ in the category of character sheaves to have a filtration by Jucys-Murphy sheaves, after we take it to the Hecke category. This object turns out to be the parabolic Springer sheaf, see Chapter 3.2 for definition, or, more precisely, its projection to the category of character sheaves. See also a related computation in the Appendix.

Recall that GL_n acts on the variety $\mathcal{N}_u$ of its unipotent elements by conjugation, and orbits of this action are numbered by partitions of n, according to the Jordan

decomposition. Let IC_λ be the intersection cohomology sheaf of the orbit numbered by a partition λ. Let λ_k be a "hook" partition $(k, 1, \dots, 1)$. We set $\lambda_n = (n)$. Let $Spr_P \simeq IC_{\lambda_1} \oplus IC_{\lambda_2}$. We have the following

Theorem (Section 4.2). *a)* $\wedge^k Spr_P \simeq (IC_{\lambda_k} \oplus IC_{\lambda_{k+1}})$, *for* $0 < k < n$.

b) $\wedge^n Spr_P \simeq IC_{\lambda_n}$

c) $\wedge^{n+1} Spr_P = 0$.

Here the exterior powers are with respect to the monoidal structure given by the convolution on GL_n.

The derived category of unipotent character sheaves is closed under convolution, but lacks a unit object. It is for this reason we must pass to the completed category, where the unit object $\hat{\delta}$ exists. Convolution with $\hat{\delta}$ gives a projection from the full equivariant derived category $D^b_{\mathrm{GL}_n}(\mathrm{GL}_n)$ of constructible sheaves on GL_n to the completed category of unipotent character sheaves. In Section 4.2 we will prove that $\wedge^n Spr_P * \hat{\delta}$ is an invertible object in the latter category, and so construct a functor from $\mathrm{Rep}(\mathrm{GL}_n)$ to it via the strategy indicated above.

1.5 Lie algebra version

Although it is not directly related to the categorification of the homomorphism in question, we also prove a variant of the theorem of the previous section in the setting of equivariant sheaves on the Lie algebra of GL_n. Here the proof, employing the notion of the Fourier transform of sheaves on vector spaces, is similar, but more direct.

Namely, identify $\mathcal{N}_u$ withe the nilpotent cone $\mathcal{N} \subset \mathfrak{gl}_\mathfrak{n} = \mathrm{Lie}(\mathrm{GL}_n)$, and again denote by IC_λ the intersection cohomology sheaves on the corresponding orbits. Let

$$\mathfrak{spr}_P = IC_{\lambda_1} \oplus IC_{\lambda_2}.$$

We have the following

Theorem (Section 4.3). *a)* $\wedge^k \mathfrak{spr}_P \simeq (IC_{\lambda_k} \oplus IC_{\lambda_{k+1}})$, *for* $0 < k < n$.

b) $\wedge^n \mathfrak{spr}_P \simeq IC_{\lambda_n}$

c) $\wedge^{n+1} \mathfrak{spr}_P = 0$.

Here the exterior powers are with respect to the monoidal structure given by the additive convolution on $\mathfrak{gl}_n$.

1.6 The functor from the perfect derived category of St

We are in the following situation: we have a monoidal functor from the Satake category $\mathrm{Rep}(\mathrm{GL}_n)$ to $\hat{D}_f$, as well as (products of) Jucys-Murphy sheaves there, categorifying the image of the Bernstein's lattice of the affine Hecke algebra. Constructions

of [Gai01] and [AB09] predict that we should have two additional pieces of data: monodromy endomorphism of objects coming from the Satake category, and a compatible filtration of these by Jucys-Murphy sheaves, corresponding to the filtration of Gaitsgory's sheaves by Wakimoto sheaves. Presence of this additional data, which we construct in Section 5 from the fact that our central sheaves can be obtained by some parabolic induction and restriction procedure, puts us in the situation of [Bez16]. In op.cit., such data was used to construct a functor from the coherent version of the affine Hecke category to its constructible version. We will employ it to construct a functor from (the perfect part of) the coherent version of the affine Hecke category to the finite Hecke category $\hat{D}_f$.

1.7 Relation to knot invariants, Hilbert schemes and matrix factorizations

There is a body of work on various geometric constructions of knot invariants, which seem to be related to the categorification of the homomorphism Π.

One of the initial motivations of this thesis project was to understand some structures arising in the recent work [GNR16]. In particular, there is the following conjecture. Let FHilb^{dg} be the flag Hilbert dg-scheme of [GNR16], and let SBim_n be the category of Soergel bimodules, see [Soe07]. Homotopy category $K^b(\mathrm{SBim}_n)$ (or, rather, its completed variant) is another categorification of the finite Hecke algebra, equivalent to the one we use, see [BY13].

Conjecture ([GNR16]). *There is a pair of adjoint functors*

$$K^b(\mathrm{SBim}_n) \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} D^b\left(Coh_{\mathbb{C}^*\times\mathbb{C}^*}\left(\mathrm{FHilb}_n^{dg}(\mathbb{C})\right)\right),$$

ι^* being monoidal and fully faithful.

In particular, functor from the conjecture above is predicted to send certain line bundles on FHilb_n^{dg} to similarly defined Jucys-Murphy objects in $K^b(\mathrm{SBim}_n)$, just as in our conjecture similar line bundles on the diagonal in St are predicted to be sent to Jucys-Murphy sheaves in $\hat{D}_f$.

In the work [OR17], a functor between two convolution algebras of equivariant matrix factorization is constructed, and is shown to intertwine the braid version of the homomorphism Π for the images of affine and finite braid groups inside the corresponding convolution algebras. This is also applied to the computation of knot homology.

At this point we do not know any direct relation of the functors constructed in this thesis to the above settings.

1.8 Organization of the paper

This thesis is organized as follows.

In Chapter 2 we describe the algebraic setting which we would like to categorify and give some supporting computations.

In Chapter 3 we define the geometric setup with which we will be working, both in constructible and coherent settings, and recall the necessary facts and constructions involving character sheaves.

In Chapter 4 we construct a functor between central categories, and discuss a variant of the main theorem for Lie algebras.

In Chapter 5 we extend this functor to a functor from the perfect derived category.

In the appendix, we make a remark about a relation of multiplicative Jucys-Murphy elements to the Kirillov-Pak character formula.

Chapter 2

Algebraic preliminaries

2.1 Hecke algebras

Let $W = S_n$ be a finite Weyl group of type A, identified with the symmetric group on n elements, let $I \subset W$ be the set of simple reflections,

$$I = \{(1\ 2), (2\ 3), ..., (n-1\ n)\}.$$

Let $l : W \to \mathbb{Z}^+$ be the length function.

The finite Hecke algebra $\mathbb{H}(W) = \mathbb{H}_n$ is a unital algebra over $\mathbb{Z}[v, v^{-1}]$ generated by elements $t_s, s \in I$, subject to the following relations. Denote $t_i = t_{(i(i+1))}$.

1. $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$.
2. $t_i t_j = t_j t_i, |i-j| > 1$.
3. $t_i^2 = 1 + (v^{-1} - v) t_i$.

$\mathbb{H}_n$ has a basis $t_w, w \in W$, defined by $t_w = t_{s_1} \dots t_{s_k}$ for a reduced expression $w = s_1 \dots s_k$, and the Kazhdan-Lusztig basis $C_w, w \in W$, see [KL79].

Let $(\mathbb{X}^*, \Phi, \mathbb{X}_*, \Phi^\vee)$ be the root datum of GL_n, namely,

$$\mathbb{X}_* = \mathbb{X}^* =: \mathbb{X} \simeq \mathbb{Z}^n = \mathrm{span}_{\mathbb{Z}}\{e_1, \dots, e_n\}, \Phi^\vee = \Phi = \{e_i - e_j\}_{i \neq j} \subset \mathbb{X}.$$

W acts on $\mathbb{X}$ and Φ permuting e_i.

Fix the set of simple roots $\Delta = \{e_i - e_{i+1}\}_{i=1}^{n-1}$. This defines the set of dominant weights to be $\mathbb{X}^+ = \{(\lambda_1, \dots, \lambda_n), \lambda_k \geq \lambda_{k+1} \text{ for all } k\}$.

The extended affine Hecke algebra $\tilde{\mathbb{H}}_n^{\mathrm{aff}}$, corresponding to this root datum, is a unital algebra over $\mathbb{Z}[v, v^{-1}]$ generated by elements $t_s, s \in I, \theta_x, x \in \mathbb{X}$, subject to the following relations. Denote $t_i = t_{(i(i+1))}, \theta_i = \theta_{e_i}$.

1. $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$.
2. $t_i t_j = t_j t_i, |i-j| > 1$.
3. $t_i^2 = 1 + (v^{-1} - v) t_i$.

4. $\theta_x\theta_y = \theta_{x+y}$.

5. $t_i\theta_j = \theta_j t_i$ if $j \neq i, i+1$.

6. $t_i\theta_i t_i = \theta_{i+1}$.

7. $\theta_0 = 1$.

For an element $\lambda \in \mathbb{X}$, let $W\lambda$ be its W-orbit.
Following [Lus83], define

$$z_\lambda = \sum_{\lambda' \in W\lambda} \theta_\lambda.$$

For a dominant $\lambda \in \mathbb{X}$ and $\mu \leq \lambda$, define $d(\mu;\lambda)$ to be the multiplicity of the weight μ in the finite-dimensional irreducible representation V_λ of GL_n with the highest weight λ. As in op.cit., define

$$s_{V_\lambda} = \sum_{\mu \leq \lambda} d(\mu;\lambda) z_\lambda.$$

We have

$$s_{V_\lambda} s_{V'_\lambda} = \sum_{\lambda''} m(\lambda,\lambda',\lambda'') s_{V''_\lambda},$$

where $m(\lambda,\lambda',\lambda'')$ satisfies

$$V_\lambda \otimes V_{\lambda'} = \oplus_{\lambda''} V_{\lambda''}^{\oplus m(\lambda,\lambda',\lambda'')}.$$

We have the following Theorem due to J. Bernstein:

Theorem ([Lus83]). *The center $\mathfrak{Z}(\tilde{\mathbb{H}}_n^{aff})$ of the extended affine Hecke algebra is a free $\mathbb{Z}[v,v^{-1}]$-module with a basis $z_\lambda, \lambda \in \mathbb{X}^+$.*

Let $\Pi : \tilde{\mathbb{H}}_n^{\mathrm{aff}} \to \mathbb{H}_n$ be the homomorphism defined by

$$\Pi(t_s) = t_s, s \in I,$$

$$\Pi(\theta_1) = 1.$$

It is easy to see, that Π is indeed a homomorphism, is defined uniquely, and that

$$\Pi(\theta_k) = t_{k-1}t_{k-2}\dots t_2 t_1^2 t_2 ... t_{k-2}t_{k-1}.$$

We denote $L_i := \Pi(\theta_i)$ and call these elements Jucys-Murphy elements in the finite Hecke algebra.

We have the following theorem:

Theorem ([DJ87],[FG06]). *Symmetric polynomials in elements L_i lie in the center $\mathfrak{Z}(\mathbb{H}_n)$ of the finite Hecke algebra. Moreover, the restriction of the homomorphism $\Pi : \mathfrak{Z}(\tilde{\mathbb{H}}_n^{aff}) \to \mathfrak{Z}(\mathbb{H}_n)$ is surjective.*

2.2 Elementary symmetric polynomials in Jucys-Murphy elements

The following interpretation of the elementary symmetric polynomials $\mathbf{e}_k$ in Jucys-Murphy elements will be useful to us. Let V_n be the standard n-dimensional representation of $\mathrm{GL}_n = \mathrm{GL}(V_n)$. We have

$$\Pi(s_{\wedge^k V_n}) = \Pi\left(\sum_{\lambda' \in W(e_1+\cdots+e_k)} \theta_{\lambda'}\right) = \mathbf{e}_k(L_1, \ldots, L_n).$$

Let $W' = S_k \times S_{n-k} \subset S_n$, considered as a parabolic subgroup with simple reflections $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_{n-1}$, and let W^k be the set of minimal length representatives of W/W'. Let $w_{0,k}$ be the longest element in S_k (with respect to the choice $\{s_1, \ldots, s_{k-1}\}$ of simple reflections). Note that $L_1 \ldots L_k = t^2_{w_{0,k}}$. We record the following straightforward computation:

Lemma 2.2.1.

$$e_k(L_1, \ldots, L_n) = \sum_{w \in W'} t_w t^2_{w_{0,k}} t_{w^{-1}}.$$

2.3 Anti-spherical projector

Let w_0 be the longest element of W,. Consider an element

$$\xi = v^{-l(w_0)} \sum_{w \in W} (-v)^{l(w)} t_w.$$

We have $\xi C_w = 0$, unless $w = 1$.

Let

$$\tilde{\mathbb{H}}^{\mathrm{aff}}_{n,perf} = \tilde{\mathbb{H}}^{\mathrm{aff}}_n \xi \tilde{\mathbb{H}}^{\mathrm{aff}}_n.$$

Let Π_{perf} be the restriction of Π to $\tilde{\mathbb{H}}^{\mathrm{aff}}_{n,perf}$. We have $\Pi_{\mathrm{perf}}(a\xi b) = \Pi(ab)\xi = c_{ab}\xi$, where $c_{ab} \in \mathbb{Z}[v, v^{-1}]$ is a coefficient of C_1 in ab.

Our main result can be considered a categorification of the projection Π_{perf}.

2.4 Universal property of $\mathrm{Rep}(\mathbf{GL}_n)$

Let $\mathcal{C}$ be a $\mathbb{C}$-linear, symmetric monoidal pseudo-abelian category. Then, for any object $X \in \mathcal{C}$ and positive integer k, we have the S_k-action on the object X^k, and, for any partition λ, Schur functors $S^\lambda X$ are defined as images of the corresponding projectors.

Lemma 2.4.1. *Let $X \in Ob(\mathcal{C})$ be such that $\wedge^n X$ is invertible and $\wedge^{n+1} X = 0$. Then there is a monoidal functor*

$$\mathrm{Rep}(GL_n) \to \mathcal{C}$$

sending the standard n-dimensional representation of $\mathrm{Rep}(GL_n)$ *to* X*.*

Proof. The fact that $\wedge^n X$ is invertible implies that X is a rigid object of dimension n. Thus, by the result of [Del07], there is a functor from the Deligne category $\mathrm{Rep}(\mathrm{GL}(t=n))$ to $\mathcal{C}$, sending the generating object to X. The fact that $\wedge^{n+1} X = 0$ implies, by the second fundamental theorem of invariant theory of the general linear group, that this functor factors through $\mathrm{Rep}(\mathrm{GL}_n)$. □

Chapter 3

Geometric setup

3.1 Sheaves on the base affine space

Contents of this section is standard, see e.g. [Lus15], [BY13].

If X is an algebraic variety with an action of an algebraic group G, let $D^b_G(X)$ be a bounded constructible G-equivariant derived category of constructible sheaves on X. If G is trivial, we write simply $D^b(X)$.

We write $\underline{\mathbb{C}}_X$ for the constant sheaf of rank 1 on X.

Let G be a connected reductive group over $\mathbb{C}$, B its Borel subgroup, $T \subset B$ the maximal torus in it, U its unipotent radical.

Let $\mathcal{B} = G/B$, let $Y = G/U$. We have a right $T \times T$-action on $Y \times Y$:

$$(t_1, t_2)(xU, yU) = (xt_1U, yt_2U).$$

Let

$$\mathcal{Y} = \frac{Y \times Y}{T},$$

where T acts on $Y \times Y$ diagonally on the right.

More generally, if $P \subset G$ is a parabolic subgroup, U_P its unipotent radical, $L \subset P$ a Levi subgroup, let $\mathcal{B}_P = G/P$, $Y_P = G/U_P$,

$$\mathcal{Y}_P = \frac{G/U_P \times G/U_P}{L}.$$

G acts on Y_P diagonally on the left.

Let $W = N(T)/T$ be the Weyl group of G, let $I \subset W$ be a set of simple reflections, let $l : W \to \mathbb{Z}^+$ be the length function.

We have the natural projection $\pi : Y \times Y \to \mathcal{B} \times \mathcal{B}$, which factors through $\mathcal{Y}$. Let $\mathcal{O}_w$ be a G-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in W$, and let $\widetilde{\mathcal{O}}_w = \pi^{-1}(\mathcal{O}_w)$. Let $\overline{\mathcal{O}}_w, \overline{\widetilde{\mathcal{O}}}_w$ be the corresponding closures, and $\widetilde{j}_w : \widetilde{\mathcal{O}}_w \to Y \times Y$ the corresponding locally-closed embedding.

Let g_w be a lift of $w \in W$ to $N(T)$. We have the projection $p_{g_w} : \widetilde{\mathcal{O}}_w \to T$, defined

by the inclusion

$$x^{-1}y \in Ug_w p_{g_w}(xU, yU)U.$$

For a local system $\mathcal{L}$ on T, let

$$\Delta^{\mathcal{L}}_{g_w} = \widetilde{j}_{w!}p^*_{g_w}\mathcal{L}[\dim T + \dim Y + l(w)], \nabla^{\mathcal{L}}_{g_w} = \widetilde{j}_{w*}p^*_{g_w}\mathcal{L}[\dim T + \dim Y + l(w)].$$

$\Delta^{\mathcal{L}}_{g_w}, \nabla^{\mathcal{L}}_{g_w}$ do not depend on a lift g_w (up to a non-canonical isomorphism). We will denote simply by $\Delta^{\mathcal{L}}_w, \nabla^{\mathcal{L}}_w$ members of their isomorphism classes.

Let $\widetilde{j}^{\bullet}_{g_w} : p_{g_w^{-1}}(1) \to Y \times Y$ be the obvious locally-closed embedding. Denote

$$\Delta^{\bullet}_{g_w} = \widetilde{j}^{\bullet}_{w!}\underline{\mathbb{C}}_{p_{g_w^{-1}}(1)}[\dim Y + l(w)].$$

Let $p_{i,j} : Y \times Y \times Y \to Y \times Y$ be a projection on i'th and j'th factors. For $\mathcal{F}, \mathcal{G} \in D^b_G(Y \times Y)$, let

$$\mathcal{F} * \mathcal{G} = p_{1,3!}(p^*_{1,2}\mathcal{F} \otimes p^*_{2,3}\mathcal{G})[-\dim Y].$$

This equips $D^b_G(Y \times Y)$ with the monoidal structure. Convolution $Y_P \times Y_P$ is defined in the same way. $D^b_G(\mathcal{Y}_P)$, considered as the subcategory of L-equivariant objects, inherits the monoidal structure.

We record the following simple

Lemma 3.1.1. *Let $\mathcal{L}$ be a W-invariant local system on T. Then $\Delta^{\mathcal{L}}_e * \Delta^{\bullet}_{g_w} \simeq \Delta^{\bullet}_{g_w} * \Delta^{\mathcal{L}}_e \simeq \Delta^{\mathcal{L}}_w$. Here $e \in W$ is the unit element.*

3.2 Harish-Chandra functor

Following [Lus85], consider the following diagram:

$$\begin{array}{ccccc} & & G \times Y_P & & \\ & \overset{\pi_G}{\swarrow} & & \overset{\widetilde{f}}{\searrow} & \\ G & & & & Y_P \times Y_P \end{array} \tag{3.1}$$

with $\pi_G(g, xP) = g, f(g, xU_P) = (xU_P, gxU_P)$, and its version obtained by taking the quotient with respect to the free right L-action:

$$\begin{array}{ccccc} & & G \times \mathcal{B}_P & & \\ & \overset{\pi_G}{\swarrow} & & \overset{f}{\searrow} & \\ G & & & & \mathcal{Y}_P \end{array} \tag{3.2}$$

with $\pi_G(g, xP) = g, f(g, xP) = (xU_P, gxU_P)$. We have functors

$$\widetilde{\mathfrak{hc}}_P = \widetilde{f}_!\pi^*_G[\dim Y_P], \widetilde{\chi}_P = \pi_{G!}\widetilde{f}^*[-\dim Y_P],$$

and

$$\mathfrak{hc}_P = f_!\pi_G^*[\dim \mathcal{B}_P], \chi_P = \pi_{G!}f^*[-\dim \mathcal{B}_P].$$

When $P = B$, we simply write

$$\mathfrak{hc}_P = \mathfrak{hc}, \chi_P = \chi.$$

We will also need intermediate functors, see [Lus04]. Let $P \subset Q$ be two parabolic subgroups.

$$\begin{array}{ccccc} & & Z_{P,Q} & & \\ & \swarrow^{\pi_Q} & & \searrow^{f} & \\ Y_Q \times Y_Q & & & & Y_P \times Y_P \end{array} \tag{3.3}$$

Here

$$Z_{P,Q} = \{(g_1U_Q, g_2U_Q, xU_P) : g_1^{-1}x \in Q\},$$

$$\widetilde{\pi}_Q(g_1U_Q, g_2U_Q, xU_P) = (g_1U_Q, g_2U_Q), \widetilde{f}(g_1U_Q, g_2U_Q, xU_P) = (xU_P, g_2g_1^{-1}xU_P).$$

Let π_Q, f be the maps obtained after dividing $\widetilde{\pi}_Q, \widetilde{f}$ by the free right L_Q, L_P actions, respectively.

Define

$$\widetilde{\mathfrak{hc}}_P^Q = \widetilde{f}_!\pi_Q^*[\dim Y_P - \dim Y_Q], \widetilde{\chi}_P^Q = \pi_{Q!}\widetilde{f}^*[\dim Y_Q - \dim Y_P],$$

and

$$\mathfrak{hc}_P^Q = f_!\pi_Q^*[\dim \mathcal{B}_P - \dim \mathcal{B}_Q], \chi_P = \pi_{Q!}f^*[\dim \mathcal{B}_Q - \dim \mathcal{B}_P].$$

By the result of op.cit., for $P \supset Q \supset R$, we have $\mathfrak{hc}_R^Q \circ \mathfrak{hc}_Q^P = \mathfrak{hc}_R^P$ and that $\mathfrak{hc}_Q^P(\mathcal{F}) = 0$ implies $\mathcal{F} = 0$.

Let $\mu : G \times G \to G$ be the multiplication morphism, let $\pi_1, \pi_2 : G \times G \to G$ be the projections on the corresponding factors. For $\mathcal{F}, \mathcal{G} \in D^b(G)$, let $\mathcal{F} * \mathcal{G} = \mu_! (\pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G})$.

Let

$$\widetilde{\mathcal{N}}_P = \{(g, xP) : g \in xU_Px^{-1}\}$$

be the parabolic Springer resolution, and let $\pi : \widetilde{\mathcal{N}}_P \to G$ be the natural projection. Define

$$Spr_P = \pi_*\mathbb{C}_{\widetilde{\mathcal{N}}_P}[2\dim U_P]$$

– the parabolic Springer sheaf.

Recall from [Gin89], that $\widetilde{\mathfrak{hc}_P}$, $\mathfrak{hc}_P$ are monoidal functors, and that we have $\widetilde{\chi_P} \circ \widetilde{\mathfrak{hc}_P}(\mathcal{F}) = \chi_P \circ \mathfrak{hc}_P(\mathcal{F}) = \mathcal{F} * Spr_P$. In particular, identity functor is a summand of $\chi_P \circ \mathfrak{hc}_P$.

Fix $J \subset I$ and let $W_J \subset W$ be the corresponding parabolic subgroup. Assume that P is conjugate to the parabolic subgroup P_J corresponding to J. Write W^J for the set of minimal length representatives of W/W_J. For every $w \in W^J$, choose a lift g_w to $N(T)$.

For an objects $X, \{C_i\}_{i=1}^k$ of a triangulated category, we write $X \in \langle C_1, C_2, ..., C_k\rangle$ if there exists a sequence of objects $\{X_i\}_{i=1}^k, X_1 = X, X_k = C_k$, such that for all

$i < k$, (C_i, X_i, X_{i+1}) is a distinguished triangle. The following lemma is a straightforward consequence of standard distinguished triangles for 6 functors for constructible sheaves.

Lemma 3.2.1. *Let X be a variety stratified by locally closed subvarieties $\{S_t\}_{t=1}^n$, and let $j_t : S_t \to X$ be the corresponding locally-closed embeddings. Assume that $S_k \subset \overline{S}_l$ implies $k > l$. Then for every $\mathcal{F} \in D^b(X)$, $\mathcal{F} \in \langle j_{t!}j_t^*\mathcal{F}\rangle_{t=1}^n$.*

The following lemma will be used to construct a filtration of certain central sheaves by Jucys-Murphy sheaves:

Lemma 3.2.2 (cf. [Gro92], [Lus13])**.** *$\widetilde{\mathfrak{hc}}(Spr_P) \in \langle \Delta^\bullet_{g_w} * \Delta^\bullet_{g_w^{-1}}\rangle_{w\in W^J}$, where objects are in some non-increasing order with respect to the Bruhat order on $\{w\}$.*

Proof. Let $\mathcal{X} = \widetilde{\mathcal{N}}_P \times Y$, and write

$$\widetilde{\phi} : \mathcal{X} \to Y \times Y, \widetilde{\phi}(g, xP, yU) = (yU, gyU).$$

By base change,

$$\widetilde{\mathfrak{hc}}(Spr_P) = \widetilde{\phi}_!\underline{\mathbb{C}}_{\mathcal{X}}[2\dim U_P].$$

Applying Lemma 3.2.1 to the filtration by

$$\mathcal{X}_w = \{(g, xP, yU) \in \mathcal{X}, x^{-1}y \in PwB\},$$

for $w \in W^J$, we get that

$$\widetilde{\mathfrak{hc}}(Spr_P) \in \langle \widetilde{\phi}_{w!}\underline{\mathbb{C}}[2\dim U_P]\rangle_{w\in W^J},$$

where $\widetilde{\phi}_w$ is a restriction $\widetilde{\phi}$ to $\mathcal{X}_w$.

We now claim that $\mathcal{X}_w$ is a bundle over the variety

$$\mathcal{C}_{g_w} = \{(xU, yU, zU) : (xU, yU) \in \widetilde{\mathcal{O}}_w, (yU, zU) \in \widetilde{\mathcal{O}}_{w^{-1}}, \\ p_{g_w}(xU, yU) = p_{g_w^{-1}}(yU, zU) = 1\},$$

with a fiber $\mathbb{A}^{\dim U_P - l(w)}$. Namely, the fibration map $\pi_{\mathcal{X}_w}$ sends $(g, xP, yU) \in \mathcal{X}_w$ to (yU, zU, gyU), where zU is a unique element of Y such that

$$(yU, zU) \in \tilde{\mathcal{O}}_w, p_{g_w}(yU, zU) = 1.$$

To check that this defines a map $\mathcal{X}_w \to \mathcal{C}_w$, we must verify that $p_{g_w^{-1}}(zU, gyU) = 1$. Indeed, we have $z^{-1}gy \in z^{-1}yU_Py^{-1}yU = z^{-1}yU$. Since $p_{g_w}(yU, zU) = 1$, we have $y^{-1}z \in Ug_wU$, so $z^{-1}y \in Ug_w^{-1}U$, and we are done.

Note that $\pi_{1,3!}\underline{\mathbb{C}}_{\mathcal{C}_w} = \Delta^\bullet_{g_w} * \Delta^\bullet_{g_w^{-1}}$, where $\pi_{1,3!}$ is the restriction of the projection $Y^3 \to Y^2$ to $\mathcal{C}_w$. We also have $\widetilde{\phi} = \pi_{1,3} \circ \pi_{\mathcal{X}_w}$.

So $\widetilde{\phi}_{w!}\underline{\mathbb{C}}_{\mathcal{X}_w}[2\dim U_P] = \Delta^\bullet_{g_w} * \Delta^\bullet_{g_w^{-1}}$. □

3.3 Parabolic restriction and induction

Let $P \subset G$ be a parabolic subgroup, L be its Levi quotient. Let

$$\tilde{G}_P = \{(g, xP) \in G \times \mathcal{B}_P : g \in xPx^{-1}\}$$

be the parabolic Grothendieck-Springer variety. Consider the standard map $\pi_L : \tilde{G}_P \to L$, and a map $f : \tilde{G}_P \to \mathcal{Y}_P$ given by the restriction of the map f from (4.1) from $G \times \mathcal{B}_P$ to $\tilde{G}_P$.

We have the monoidal functor $i_L : D^b_L(L) \to D^b_G(\mathcal{Y}_P), i_L = f_! \pi_L^*[\dim \mathcal{B}_P]$. In fact, it gives an equivalence from $D^b_L(L)$ to the subcategory of complexes in $D^b_G(\mathcal{Y}_P)$ supported on the closed subvariety $D_P = \{(xU_P, yU_P)L : xP = yP\} \subset \mathcal{Y}_P$. Let i°_L be the inverse equivalence. Let $\delta_P : D_P \to \mathcal{Y}_P$ be the corresponding closed embedding. We have the following description of the parabolic restriction and induction functors, first defined in [Lus85]:

$$\mathrm{Res}^G_L(\mathcal{F}) = i^\circ_L \delta_P^* \mathfrak{hc}_P(\mathcal{F}), \mathrm{Ind}^G_L(\mathcal{G}) = \chi_P i_L(\mathcal{G}).$$

We will also use another description of induction and restriction functors, also due to Lusztig. See also [Gin93].

We have that $\tilde{G}_P$ is a G-equivariant fiber bundle on G/P with a fiber P. The restriction to the fiber gives a bijective correspondence between G-equivariant complexes on G_P and ad-P-equivariant complexes on P. If we equip $D^b_G(\tilde{G}_P)$ with a !-convolution structure along G, it is easy to see that the above correspondence becomes monoidal. For an object $\mathcal{A} \in D^b_P(P)$ write $\tilde{\mathcal{A}}$ for the corresponding object in $D^b_G(\tilde{G}_P)$.

Let $\iota_P : P \to G$ be the corresponding closed embedding, let $q_L : P \to L, \tilde{\pi} : \tilde{G}_P \to G$ be the natural projections. We have

$$\mathrm{Res}^G_L \mathcal{A} = q_{L!} \iota_P^* \mathcal{A} = q_{L*} \iota_P^! \mathcal{A},$$

$$\mathrm{Ind}^G_L \mathcal{A} = \tilde{\pi}_* \widetilde{q_L^* \mathcal{A}}[2 \dim U_P].$$

3.4 Constructible finite Hecke category and character sheaves

$Y \times Y$ is a $T \times T$-torsor over $\mathcal{B} \times \mathcal{B}$. Thus it makes sense to talk about $T \times T$-monodromic sheaves on $Y \times Y$. For a coset $\lambda + X$ of a weight λ of $\mathrm{Lie}\, T$, let $\mathcal{P}^\lambda$ be the category of G-equivariant $T \times T$-monodromic perverse sheaves on $Y \times Y$ with generalized monodromy $\lambda \times -\lambda$. If $\lambda = 0$ we will call this category unipotent and denote simply by $\mathcal{P}$. Let $D_f = D^b\mathcal{P}$.

Our version of the Hecke category is the free-monodromic constructible derived category

$$\hat{D}_f := \hat{D}^b_c(T \times T \backslash\backslash Y \times Y)$$

defined in [BY13]. It has a perverse t-structure of its own, with the heart $\hat{\mathcal{P}} \supset \mathcal{P}$, and one has $\hat{D}_f = D^b(\hat{\mathcal{P}})$.

Remark. Note that both D_f and $\hat{D}_f$ are monoidal categories. However, D_f has no unit, while $\hat{D}_f$ has one, see below. It is for this reason we choose to work with $\hat{D}_f$. Our results could also be reformulated for certain finite quotients of pro-unipotent sheaves involved, bearing the just mentioned defect of the finite category in mind.

$\hat{D}_f$ has the subcategory $\hat{\mathcal{T}}$ of free-monodromic tilting objects. Convolution with such an object is an exact functor. It is shown in op.cit. that

$$\hat{D}_f \simeq Ho(\hat{\mathcal{T}}).$$

There is a collection $\hat{\mathcal{T}}_w, w \in W$, of indecomposable free-monodromic tilting sheaves. Moreover, $\hat{\mathcal{T}}_{w_0}$ satisfies the following property:

$$\Delta_w^{\hat{\mathcal{L}}_1} * \hat{\mathcal{T}}_{w_0} \simeq \hat{\mathcal{T}}_{w_0} * \Delta_w^{\hat{\mathcal{L}}_1} \simeq \hat{\mathcal{T}}_{w_0}.$$

The category of unipotent character sheaves, $\mathcal{CS}$, is the category of G-equivariant perverse sheaves $\mathcal{F}$ on G such that $\mathfrak{hc}(\mathcal{F}) \in D^b\mathcal{P}$. By the result of [MV88], this definition is equivalent to the standard one.

Similarly, we can consider completed categories $\hat{\mathcal{CS}}, \hat{D}_G = D^b(\hat{\mathcal{CS}})$.

Let $\hat{\mathcal{L}}_1$ be the free pro-unipotent local system on T, see [BY13]. A sheaf $\Delta_e^{\hat{\mathcal{L}}_1}$ on $\mathcal{Y}$ is the monoidal unit in the category of $D_G^b(\mathcal{Y})$. It is the image under $\mathfrak{hc}$ of the perverse sheaf $\hat{\delta} \in \hat{\mathcal{CS}}$. $\hat{\delta}$ is the monoidal (pro)-unit in the category of unipotent character sheaves on G.

We have the following

Lemma 3.4.1 ([BY13])**.**

a) $\Delta_w^{\hat{\mathcal{L}}_1} * \Delta_{w'}^{\hat{\mathcal{L}}_1} \simeq \Delta_{ww'}^{\hat{\mathcal{L}}_1}, \nabla_w^{\hat{\mathcal{L}}_1} * \nabla_{w'}^{\hat{\mathcal{L}}_1} \simeq \nabla_{ww'}^{\hat{\mathcal{L}}_1}$ *if* $l(w) + l(w') = l(ww')$.

b) $\Delta_w^{\hat{\mathcal{L}}_1} * \nabla_{w^{-1}}^{\hat{\mathcal{L}}_1} \simeq \nabla_w^{\hat{\mathcal{L}}_1} * \Delta_{w^{-1}}^{\hat{\mathcal{L}}_1} \simeq \Delta_e^{\hat{\mathcal{L}}_1}$.

c) $\mathbb{D}\Delta_w^{\hat{\mathcal{L}}_1} \simeq \nabla_w^{\hat{\mathcal{L}}_1}$.

In the same way, if $P \supset B$ is a parabolic subgroup with a Levi subgroup L, and λ is a weight fixed by $W(L)$, we have the pro-local system on T of "infinite Jordan type" λ, $\hat{\mathcal{L}}_\lambda$, see [Gin88], and a corresponding sheaf $\Delta_e^{\hat{\mathcal{L}}_\lambda}$. It is an image under $\mathfrak{hc}_B^P$ of the parabolic character sheaf $\hat{\delta}_\lambda$ on $\mathcal{Y}_P$.

When we say that some object X is invertible with respect to the convolution in the category $\hat{D}_f$ (or in the category of unipotent character sheaves), we mean that there is an object Y in the corresponding category, such that

$$X * Y \simeq Y * X \simeq \Delta_e^{\hat{\mathcal{L}}_1}$$

$$(X * Y \simeq Y * X \simeq \hat{\delta}, \text{ respectively}).$$

3.5 Braided structure and the twist

Here we recall the necessary constructions from [BD14].

We will need a notion of the braided structure on the G-equivariant derived category of constructible sheaves on G.

Define $\xi : G \times G \to G \times G, (g,h) \mapsto (g, g^{-1}hg), \tau : G \times G \to G \times G, (g,h) \mapsto (h,g)$. We have $\mu \circ \xi = \mu \circ \tau$.

For $A, B \in D^b_G(G)$, we have $A \boxtimes B \simeq \xi_! (A \boxtimes B)$ by G-equivariance of B. So

$$A * B = \mu_!(A \boxtimes B) \simeq \mu_! \xi_! (A \boxtimes B) = \mu_! \tau_! (A \boxtimes B) \simeq B * A.$$

This defines a braided structure on $D^b_G(G)$, denoted by

$$\beta_{A,B} : A * B \to B * A.$$

For any $M \in D^b_G(G)$ we have an isomorphism $\theta_M : M \to M$ called a twist, see op.cit. It satisfies $\theta_{A*B} = \beta_{A,B} \circ \beta_{B,A} \circ (\theta_A * \theta_B)$. On the stalk of M at a point $g \in G$ it acts by the conjugation by $g \in Z_G(g)$ – centralizer of g in G.

Corollary 3.5.1. *Assume that centralizers of all points in G are connected. Then $D^b_G(G)$ is symmetric, meaning that*

$$\beta_{A,B} \circ \beta_{B,A} = Id_{B*A}$$

for all $A, B \in D^b_G(G)$.

3.6 Sheaves on tori

We recall several facts about the convolution of constructible sheaves on tori, see [GL96].

For a weight λ of $\operatorname{Lie} T$ let $\mathcal{L}_\lambda$ be the corresponding local system on T.

Lemma 3.6.1. *Let $\mu^\vee : \mathbb{G}_m \to T$ be a coweight of T such that $\langle \lambda, \mu^\vee \rangle \notin \mathbb{Z}$. Then, for any constructible sheaf $\mathcal{F}$ on T, equivariant with respect to the $\mathbb{G}_m$-action defined by $\mu^\vee$, we have $\mathcal{L}_\lambda * \mathcal{F} = 0$.*

Proof. It is enough to consider the case $T = \mathbb{G}_m$, in which case the statement is equivalent to the fact $H^*_c(\mathcal{L}_\lambda) = 0$ for $\lambda \neq 0 \in \mathbb{C}/2\pi i\mathbb{Z}$. □

Lemma 3.6.2. *Let $\mathcal{F} \in D^b(T)$ be such that $\mathcal{F} * \mathcal{L}_\lambda = 0$ for all $\lambda \neq 0$. Then for all i, perverse cohomology sheaves ${}^pH^i\mathcal{F}$ are shifted local systems on T with unipotent monodromy.*

Our proof is a direct adaptation of the proof of Proposition 3.4.5 [GL96]. We state this proposition and another lemma from op.cit., originally due to Laumon.

Lemma 3.6.3. *Let $\mathcal{F} \in D^b(T)$ on T such that $H^*_c(\mathcal{F} \otimes \mathcal{L}_\lambda) = 0$ for all λ. Then $\mathcal{F} = 0$.*

Proof. This is Proposition 3.4.5 in op.cit. □

Lemma 3.6.4. *Let $\mathcal{F} \in D^b(T)$ be perverse. Then*

$$\chi(\mathcal{F}) = \sum_i (-1)^i H_c^i(\mathcal{F}) \geq 0.$$

Proof. This is Corollary 3.4.4 in op.cit. □

Proof of Lemma 3.6.2. First note that the condition $\mathcal{F} * \mathcal{L}_\lambda = 0$ is equivalent to the condition $H_c^*(\mathcal{F} \otimes \mathcal{L}_\lambda) = 0$. Assume that the statement is known for tori of rank $< n$, and let $n = \operatorname{rk} T$. Let $T = T' \times T''$, where T', T'' have smaller ranks. Then, for any weight (λ', λ'') of $T' \times T''$, projection formula gives

$$\pi_{T'!}(\mathcal{F} \otimes \mathcal{L}_{(\lambda', \lambda'')}) \simeq \pi_{T'!}(\mathcal{F} \otimes \mathcal{L}_{(0, \lambda'')}) \otimes \mathcal{L}_{\lambda'}.$$

So, if $\lambda'' \neq 0$, we get, by Lemma 3.6.3, that $\pi_{T'!}(\mathcal{F} \otimes \mathcal{L}_{(0,\lambda'')}) = 0$, so that $\mathcal{F}$ is smooth with unipotent monodromy along all fibers of $\pi_{T'}$. Since T' was chosen arbitrarily, this implies that $\mathcal{F}$ is smooth with unipotent monodromy.

We now need to prove that the result holds for $\mathbb{G}_m$. Consider the lowest non-zero perverse cohomology sheaf $A = {}^pH^{min}(\mathcal{F})$. Let $A' \subset A$ be an irreducible subsheaf. Since $H_c^0(\cdot)$ is t-exact from the left, we get that $H_c^0(A' \otimes \mathcal{L}_\lambda) = 0$. This means that A' can not be a punctual sheaf, so it is a shift of a (non-derived) $*$-extension of some local system on a Zariski open subset $U \subset \mathbb{G}_m$. We also have $\chi(A' \otimes \mathcal{L}_\lambda) = H_c^0(A' \otimes \mathcal{L}_\lambda) - H_c^1(A' \otimes \mathcal{L}_\lambda) \leq 0$. By Lemma 3.6.4, this implies that $\chi(A') = 0$, so, by the Euler-Poincare formula, A' is either 0 or a shifted rank one local system on $\mathbb{G}_m$. Since $H_c^0(A' \otimes \mathcal{L}_\lambda) = 0$ for $\lambda \neq 0$, this implies that A' is $\underline{\mathbb{C}}_{\mathbb{G}_m}[1]$. Then $H_c^0(A/A' \otimes \mathcal{L}_\lambda) = 0$, and we can repeat the argument by induction on length of A to get that A is a shifted unipotent local system. Again, this implies that $H_c^*(\tau^{\geq min+1}\mathcal{F} \otimes \mathcal{L}_\lambda) = 0$, and we're done by induction.

□

Corollary 3.6.1. *Assume that for $\mathcal{F} \in D_G^b(G)$ we have $\mathfrak{hc}(\mathcal{F}) * \Delta_e^{\hat{\mathcal{L}}_\lambda} = 0$ for every $\lambda \neq 0$. Then $\mathcal{F} \in D^b(\mathcal{CS})$.*

3.7 Affine Hecke categories

Here we explain the relations between various variants of the affine Hecke category.

Let

$$\mathcal{K} = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]].$$

Let I be an Iwahori subgroup of $\mathrm{GL}_n(\mathcal{K})$, let I^0 be its pro-unipotent radical. Let $\mathrm{Fl}^{\mathrm{aff}} = \mathrm{GL}_n(\mathcal{K})/I$ be the affine flag variety.

Let $\widetilde{\mathrm{Fl}}^{\mathrm{aff}} = \mathrm{GL}_n(\mathcal{K})/I^0$. We have $I/I^0 = T$ and $\widetilde{\mathrm{Fl}}^{\mathrm{aff}}$ is a T-torsor over $\mathrm{Fl}^{\mathrm{aff}}$ with respect to the right T-action.

Let $D_{I^0 I^0}$ be the full subcategory of I^0-equivariant sheaves on $\widetilde{\mathrm{Fl}}^{\mathrm{aff}}$ consisting of complexes whose cohomology is monodromic with respect to the above T-torsor structure.

Let $\hat{D}$ be the pro-completion of the category $D_{I^0 I^0}$ defined in [BY13].

Both categories $\hat{D}$ and $D_{I^0 I^0}$ carry a monoidal structure. Remark of Section 3.4 applies here equally.

Recall from [Bez16] that we have the spectral variant of the affine Hecke category. For an algebraic variety X, let $Coh(X)$ be the category of coherent sheaves on X.

For a closed subscheme $Z \subset X$, let $Coh_Z(X)$ be the full subcategory of $Coh(X)$ consisting of sheaves set-theoretically supported on Z.

If X is equipped with an action of an algebraic group G, let $Coh^G(X)$ be the category of G-equivariant coherent sheaves on X, and let $D^G_{\mathrm{perf}}(X) \subset D^b(Coh^G(X))$ be the subcategory of perfect complexes.

Let $\mathfrak{g}^\check{}$ be the Lie algebra of the Langlands dual group $G^\check{}$, let $\tilde{\mathfrak{g}}^\check{}$ be the corresponding Grothendieck-Springer resolution, $\mathrm{St} = \tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}$.

Let $\mathcal{N} \subset \mathfrak{g}^\check{}$ be the nilpotent cone.

Let $\hat{\mathfrak{g}}^\check{}$ be the spectrum of the completion of the ring of functions $\mathcal{O}(\mathfrak{g}^\check{})$ at $\mathcal{N}$.

We have a completed variant of $\mathrm{St}, \hat{\mathrm{St}} = \mathrm{St} \times_{\mathfrak{g}^\check{}} \hat{\mathfrak{g}}^\check{}$.

Categories $D^b(Coh^G_{\mathcal{N}}(\mathrm{St})), D^b(Coh^G(\hat{\mathrm{St}}))$ have a natural monoidal structure given by the convolution.

We have the following

Theorem ([Bez16])**.** *The following are equivalent as monoidal categories:*

$$D^b(Coh^{G^\check{}}_{\mathcal{N}^\check{}}(St)) \simeq D_{I^0 I^0},$$

$$D^b(Coh^{G^\check{}}(\hat{St})) \simeq \hat{D}.$$

It is explained in op.cit. that the category $D^{G^\check{}}_{\mathrm{perf}}(\hat{\mathrm{St}})$ can be thought of as a categorification of the perfect subalgebra $\tilde{\mathbb{H}}^{\mathrm{aff}}_{\mathrm{perf}}$, see Section 2.3.

We have two natural $G^\check{}$-equivariant projections $\pi_{1,2} : \mathrm{St} \to \mathcal{B}$, and, for any two weights λ_1, λ_2 of T and corresponding equivariant line bundles $\mathcal{O}(\lambda_1), \mathcal{O}(\lambda_2)$ on $\mathcal{B}$, we have a line bundle $\mathcal{O}_{\hat{\mathrm{St}}}(\lambda_1, \lambda_2) = \pi_1^* \mathcal{O}(\lambda_1) \otimes_{\mathcal{O}_{\mathrm{St}}} \pi_2^* \mathcal{O}(\lambda_2) \in D^G_{\mathrm{perf}}(\hat{\mathrm{St}})$.

For a representation V of $G^\check{}$, we have a corresponding $G^\check{}$-equivariant bundle $V \otimes \mathcal{O}_{\hat{\mathrm{St}}}$.

Chapter 4

Construction of the functor from $\mathrm{Rep}(\mathbf{GL}_n)$

4.1 Sheaves on the unipotent variety

In this section $G = \mathrm{GL}_n$.

Let V_n be the standard n-dimensional representation of $\mathrm{GL}_n = \mathrm{GL}(V_n)$. Let P be a parabolic subgroup of GL_n fixing a vector in $\mathbb{P}(V_n)$.

Recall that G acts on the variety $\mathcal{N}_u$ of unipotent elements in G by conjugation, and orbits of this action are numbered by partitions of n, according to the Jordan decomposition. Let IC_λ be the intersection cohomology sheaf of the orbit numbered by a partition λ. Let λ_k be a "hook" partition $(k, 1, \ldots, 1)$, and let $\lambda_n = (n)$. Then $Spr_P \simeq IC_{\lambda_1} \oplus IC_{\lambda_2}$. Convolutions $IC_\lambda * \hat{\delta}$ are pro-objects of the category $\mathcal{CS}$.

4.2 Main theorem

By Corollary 3.5.1, the category $D^b_G(G)$ is a symmetric monoidal category. This allows us to define for any object $A \in D^b_G(G)$ objects $\wedge^k A$. We are ready to state the main theorem of this Chapter:

Theorem 1.

a) $\wedge^k Spr_P \simeq (IC_{\lambda_k} \oplus IC_{\lambda_{k+1}})$, *for* $0 < k < n$.

b) $\wedge^n Spr_P \simeq IC_{\lambda_n}$

c) $IC_{\lambda_n} * \hat{\delta}$ *is invertible under convolution.*

d) $\wedge^{n+1} Spr_P = 0$.

Corollary 4.2.1. *There is a monoidal functor* $\varpi_Z : \mathrm{Rep}(G) \to \hat{D}_G$, *sending* V_n *to* $Spr_P * \hat{\delta}$, *and a monoidal functor*

$$\mathfrak{hc} \circ \varpi_Z : \mathrm{Rep}(G) \to \hat{D}_f,$$

sending representations of GL_n to central objects.

The proof will proceed by induction in n. When $n = 1$, $Spr_P = \delta_1$ – a sheaf supported on $1 \in \mathbb{C}^*$ with a 1-demensional stalk, and we need only the fact that $\wedge^2\delta_1 = 0$, which is obvious. Assume that the theorem is known for groups $\mathrm{GL}_k, k = 1, \ldots, n-1$.

Proof of a). We will now compute $\wedge^k Spr_P$ from the known answer for GL_k. Let $P_k \supset B$ be a parabolic subgroup of operators fixing a given $k-$dimensional subspace $V_k \subset V_n$, and let $L_k \simeq \mathrm{GL}_k \times \mathrm{GL}_{n-k}$ be its Levi subgroup, regarded as a two-block subgroup of GL_n.

Let $G' = \mathrm{GL}_k = \mathrm{GL}(V_k)$, let P' be a parabolic subgroup fixing a vector in $\mathbb{P}(V_k)$. We have a corresponding parabolic Springer sheaf $Spr_{P'}$ on G'. Let $Spr_k = Spr_{P'} \boxtimes \delta_1$, a perverse sheaf on L_k.

Recall that $\pi_{P_k} : P_k \to L_k$ denotes the standard projection.

For a line $l = xP \in G/P \simeq \mathbb{P}(V_n)$, let $U_l = xU_Px^{-1}$.

Consider the following variety:

$$\tilde{\mathcal{N}}'_P = \{(l, g) \in \mathbb{P}(V_k) \times \mathrm{GL}_n : g \in U_l\}.$$

We have the natural projection $\pi_{\tilde{\mathcal{N}}'_P} : \widetilde{\mathcal{N}}'_P \to P_k$. Define

$$\mathcal{F}_1 = \pi_{\tilde{\mathcal{N}}'_P *} \underline{\mathbb{C}}_{\tilde{\mathcal{N}}'_P}[2 \dim U_P],$$

$$\mathcal{F}_2 = \pi^*_{P_k} Spr_k[2 \dim U_{P_k}].$$

Note that, since centralizers of all elements in P_k are also connected, it makes sense to talk about Schur functors with respect to the convolution in the category $D^b_{P_k}(P_k)$, see Corollary 3.5.1.

Recall notations from Section 3.3.

Lemma 4.2.1. $\tilde{\pi}_* \widetilde{\wedge^k \mathcal{F}_1} \simeq \wedge^k Spr_P$.

Proof. We will regard elements of G/P_k as elements of the Grassmannian $Gr(k, n)$ of k-dimensional subspaces in V.

We have the following convolution varieties for $\tilde{\pi}_* \widetilde{\mathcal{F}_1^{*k}}$ and Spr_P^k:

$$\tilde{C}_1^k = \{(l_1, \ldots, l_k, g_1, \ldots, g_k, V') \in \mathbb{P}(V_n)^k \times G^k \times G/P_k : g_i \in U_{l_i}, l_i \subset V'\},$$

$$C^k = \{(l_1, \ldots, l_k, g_1, \ldots, g_k) \in \mathbb{P}(V_n)^k \times G^k : g_i \in U_{l_i}\},$$

and a commutative diagram

$$\begin{array}{ccc} \tilde{C}_1^k & \xrightarrow{f} & C^k \\ & \searrow^{m_k} & \downarrow m_k \\ & & G \end{array} \tag{4.1}$$

Here f is the map forgetting the k-subspace V', and m_k is the composition of the projection to G^k and multiplication $(g_1, \ldots, g_k) \mapsto g_1 \ldots g_k$.

We have

$$\tilde{\pi}_*\widetilde{\mathcal{F}_1^{*k}} \simeq m_{k!}\underline{\mathbb{C}}_{\tilde{C}_1^k}[2k\dim U_P] \simeq m_{k!}f_!\underline{\mathbb{C}}_{\tilde{C}_1^k}[2k\dim U_P],$$

$$Spr_P^k \simeq m_{k!}\underline{\mathbb{C}}_{C^k}[2k\dim U_P]$$

.

Since f is proper and an isomorphism over an open set where $\{l_1,\dots,l_k\}$ are in general position, $f_!\underline{\mathbb{C}}_{C_2^k}$ is a semi-simple complex, containing $\underline{\mathbb{C}}_{\tilde{C}_1^k}$ as a direct summand. We get that

$$\wedge^k Spr_P \Subset \tilde{\pi}_*\widetilde{\wedge^k\mathcal{F}_1} \simeq IC_{\lambda_k} \oplus IC_{\lambda_{k+1}}.$$

Now, since hypercohomology with compact support is a monoidal functor, we can compute it for $\wedge^k Spr_P$ and get that the hypercohomologies of the sides of the above inclusion are the same, so it is an isomorphism. □

Lemma 4.2.2. *a)* $\wedge^k\mathcal{F}_1 \simeq \wedge^k\mathcal{F}_2$.

b) $\wedge^{k+1}\mathcal{F}_1 = 0$.

Proof. Assume that we know that the lemma is proven for all $k_0 < k$. We will now prove it for k.

It is easy to see that U_{P_k}-averaging of $\mathcal{F}_1$ and $\mathcal{F}_2$ is the same, namely

$$\pi_{P_k!}\mathcal{F}_1 \simeq \pi_{P_k!}\mathcal{F}_2,$$

and, since $\pi_{P_k!}$ is monoidal,

$$\pi_{P_k!}\wedge^k\mathcal{F}_1 \simeq \pi_{P_k!}\wedge^k\mathcal{F}_2.$$

Since $\mathcal{F}_2$ is U_{P_k}-equivariant, we have

$$\wedge^k\mathcal{F}_2 \simeq \pi_{P_k}^*\pi_{P_k!}\wedge^k\mathcal{F}_2[2\dim U_Q]$$

Consider a variety

$$C_1^k = \{(l_1,\dots,l_k,g_1,\dots,g_k) \in \mathbb{P}(V_k)^k \times G^k : g_i \in U_{l_i}\}.$$

We have a morphism

$$m_k : C_1^k \to P_k, (l_1,\dots,l_k,g_1,\dots,g_k) \mapsto g_1\dots g_k,$$

and $m_{k!}\underline{\mathbb{C}}_{C_1^k}[2k\dim U_P] = \mathcal{F}_1^{*k}$. Consider an open subset $\mathcal{U}^k$ of C_1^k defined as a subset of $(l_1,\dots,l_k,g_1\dots,g_k)$ with $(l_1,\dots,l_k)$ in general position, and let Z^k be its complement in C_1^k. We denote by m_k, abusing notation, the restriction of m_k above to $\mathcal{U}^k, Z^k$. We have a distinguished triangle

$$(m_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k}, \mathcal{F}_1^{*k}, m_{k!}\underline{\mathbb{C}}_{Z^k}). \tag{4.2}$$

It is easy to see that the complexes $m_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k}, m_{k!}\underline{\mathbb{C}}_{Z^k}$ inherit the S_k-action, compatible with the S_k-action on $\mathcal{F}_1^{*k}$ coming from the braiding.

We now claim that $m_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k}$ is U_{P_k}-equivariant. For this purpose, we construct a map α, making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{U}^k \times U_{P_k} & \xrightarrow{\alpha} & \mathcal{U}^k \\ & \searrow^{m'_k} & \downarrow{\scriptstyle m_k} \\ & & P_k \end{array}$$

Here m'_k is the composition

$$\mathcal{U}^k \times U_{P_k} \xrightarrow{m_k \times Id} P_k \times U_{P_k} \xrightarrow{(x,u)\mapsto xu} P_k,$$

Choice of a Levi subgroup L_k of P_k gives a splitting $V = V_k \oplus W$. Given such a splitting, U_{P_k} is identified with the subgroup $Id + Hom(W, V_k)$. Any splitting

$$V_k = l'_1 \oplus \cdots \oplus l'_k, \dim l'_i = 1,$$

defines a splitting

$$U_Q = (Id + Hom(W, l'_1)) \times \cdots \times (Id + Hom(W, l'_k)),$$

with $(Id + Hom(W, l'_i)) \subset U_{l'_i}$.

Pick a point $(l_1, \ldots, l_k, g_1 \ldots, g_k, u) \in \mathcal{U}^k \times U_{P_k}$. Let

$$R_i = g_{i+1} g_{i+2} \ldots g_k,$$

$$l'_i = R_i^{-1} l_i.$$

Note that, since (l_i) are in general position, so are (l'_i), since the transition matrix in PGL_k between these collections is triangular. Let

$$u = u'_1 \ldots u'_k, u'_i \in U_{l'_i}$$

be a splitting of u corresponding to the splitting (l'_i) of V_k, as above. Let

$$u_i = R_i u'_i R_i^{-1}.$$

Note that $u_i \in U_{l_i} = U_{R_i l'_i}$. We set

$$\alpha(l_1, \ldots, l_k, g_1, \ldots, g_k, u) = (l_1, \ldots, l_k, g_1 u_1, \ldots, g_k u_k).$$

It easy to see that $g_1 \ldots g_k u = g_1 u_1 \ldots g_k u_k$, so the desired map α is now constructed.

It remains to observe that α defines a free action of U_{P_k} on $\mathcal{U}^k$, so that we have

$$m'_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k \times U_{P_k}}[2 \dim U_{P_k}] \simeq m_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k},$$

so the latter sheaf is indeed U_{P_k}-equivariant.

Recall the distinguished triangle (4.2). Applying the projection ψ to the isotypical component of the sign representation to it, we get a triangle

$$(\psi m_{k!}\underline{\mathbb{C}}_{\mathcal{U}^k}, \wedge^k\mathcal{F}_1, \psi m_{k!}\underline{\mathbb{C}}_{Z^k}).$$

First complex in this triangle is U_{P_k}-equivariant, so, to prove part a) of the lemma, it is enough to show that $\psi m_{k!}\underline{\mathbb{C}}_{Z^k} \simeq 0$. But this follows from the part b) by the inductive assumption for $k_0 = k-1$, definition of Z_k and the fact that $m_{k!}\underline{\mathbb{C}}_{Z^k}$ is a direct summand (of the symmetrization of, as in Section 3.3) of the corresponding exterior power on P_{k_0}, as in the proof of Lemma 4.2.1.

To prove part b), note that $\wedge^{k+1}\mathcal{F}_1$ is a direct summand of $\wedge^k\mathcal{F}_1 * \mathcal{F}_1$. Since the first factor is U_{P_k}-equivariant, so is the whole convolution and

$$\wedge^{k+1}\mathcal{F}_1 \simeq \wedge^{k+1}\mathcal{F}_2 \simeq 0.$$

□

By our inductive assumption, we want to prove that $\wedge^k Spr_P \simeq \mathrm{Ind}_{L_k}^G(\wedge^k Spr_k)$, which follows from the above two lemmas. □

For a parabolic subgroup $Q \supset B$ with a Levi L, we say that a weight λ of $\mathrm{Lie}\, T$ is Q-regular, if its stabilizer in the full Weyl group W is equal to $W(L)$. Lemma below and its use in the proof are due to Roman Bezrukavnikov.

Lemma 4.2.3. *Let the weight λ of T be Q-regular. Then for any $\mathcal{F} \in D_G^b(G)$,*

$$\mathfrak{hc}_Q(\mathcal{F}) * \hat{\delta}_\lambda \simeq (i_L \circ \mathrm{Res}_L^G(\mathcal{F})) * \hat{\delta}_\lambda.$$

Proof. (see Section 3.3 for notations)

By definition, we have $i_L \circ \mathrm{Res}_L^G(\mathcal{F}) = \delta_{Q*}\delta_Q^*\mathfrak{hc}_Q(\mathcal{F})$. Let $j : \mathcal{Y}_Q \backslash D_Q \to \mathcal{Y}_Q$ be the complementary open embedding. We have a distinguished triangle

$$(j_!j^*\mathfrak{hc}_Q(\mathcal{F}), \mathfrak{hc}_Q(\mathcal{F}), i_L \circ \mathrm{Res}_L^G(\mathcal{F})).$$

We will now prove that $j_!j^*\mathfrak{hc}_Q(\mathcal{F}) * \hat{\delta}_\lambda = 0$. We have

$$\mathfrak{hc}_B^Q(j_!j^*\mathfrak{hc}_Q(\mathcal{F}) * \hat{\delta}_\lambda) = j'_!\mathcal{G} * \Delta_e^{\hat{\mathcal{L}}_\lambda},$$

where j' is an embedding of an open subset $V = \{(xU, yU)T \in \mathcal{Y}, x^{-1}y \notin Q\}$, and $\mathcal{G} \in D_G^b(V)$. Let $\tilde{V}$ be the preimage of V in $Y \times Y$. $\tilde{V}$ has a filtration by locally closed subsets $\tilde{V}_w = \{(xU, yU), x^{-1}y \in Ug_wQ\}$, where w runs through W^J for a subset J of simple reflections in W, corresponding to Q, and g_w are some lifts of w to $Z_G(T)$. Let $\tilde{j}'_w : \tilde{V}_w \to \tilde{V}$ be the corresponding locally closed embeddings. Consider $j'_!\mathcal{G}$ as a T-equivariant complex on $Y \times Y$. By Lemma 3.2.1, $j'_!\mathcal{G} \in \langle \tilde{j}'_{w!}\tilde{j}'^*_w j'_!\mathcal{G}\rangle_{w \in W^J}$, so it is enough to prove that $\tilde{j}'_{w!}\tilde{j}'^*_w j'_!\mathcal{G} * \Delta_e^{\hat{\mathcal{L}}_\lambda} = 0$. Consider a point $(xU, yU) \in \tilde{V}_w$. Choose

x, y in the corresponding congruence classes, so that $x^{-1}y = g_w l$, $l \in L$. Since λ is Q-regular, there is a coweight $\mu^\vee : \mathbb{G}_m \to T$ with $\langle \lambda, \mu^\vee - w(\mu^\vee)\rangle \neq 0$, and such that $\mu^\vee(\mathbb{G}_m)$ is in the center of L. Then, for $t \in \mathbb{G}_m$, we have

$$\mathrm{Ad}(\mu^\vee(t))(x^{-1}y) = \mu^\vee(t) g_w l \mu^\vee(t)^{-1} = \mu^\vee(t) w(\mu^\vee(t))^{-1}(x^{-1}y),$$

so that the G-equivariant sheaf, equivariant with respect to the right diagonal T-action, is also equivariant with respect to the action of $\mathbb{G}_m$ defined by the coweight $(\mu^\vee - w(\mu^\vee))$ on one of the components. By Lemma 3.6.1, we get the result. □

Note that since $Spr_P = IC_{\lambda_1} \oplus IC_{\lambda_2}$, we have

$$\wedge^k Spr_P \simeq \wedge^k IC_{\lambda_2} \oplus \wedge^{k-1} IC_{\lambda_2}.$$

Proof of c). By the result of [BFO12], the functor $\nabla^{\hat{\mathcal{L}}_1}_{w_0} * \mathfrak{hc}(\cdot)$ commutes with the Verdier duality. We get that

$$\Delta^{\hat{\mathcal{L}}_1}_{w_0} \simeq \mathbb{D}(\nabla^{\hat{\mathcal{L}}_1}_{w_0} * \mathfrak{hc}(\hat{\delta})) \simeq \nabla^{\hat{\mathcal{L}}_1}_{w_0} * \mathfrak{hc}(\mathbb{D}\hat{\delta}),$$

so that $\mathfrak{hc}(\mathbb{D}\hat{\delta}) \simeq \Delta^{\hat{\mathcal{L}}_1}_{w_0} * \Delta^{\hat{\mathcal{L}}_1}_{w_0}$, by Lemma 3.4.1, which is invertible. But $\hat{\delta}$ is the IC-extension of the shifted local system $\mathcal{E}$ on a set of regular semisimple elements of G, dual of which is isomorphic to $\mathrm{sgn} \otimes \mathcal{E}$, where sgn is the local system corresponding to the sign representation of S_n. It is easy to see that IC-extension of $\mathrm{sgn} \otimes \mathcal{E}$ is $IC_{\lambda_n} * \hat{\delta}$, so the latter is invertible. □

Proof of d). This is equivalent to $\wedge^n IC_{\lambda_2} = 0$. By induction and by Lemma 4.2, we get that $\mathfrak{hc}(\wedge^n IC_{\lambda_2}) * \Delta^{\hat{\mathcal{L}}_\lambda}_e \simeq 0$ for all $\lambda \neq 0$. So, by Corollary 3.6.1, $\wedge^n IC_{\lambda_2}$ is a unipotent character sheaf. On the other hand,

$$\wedge^n IC_{\lambda_2} \simeq \wedge^n IC_{\lambda_2} * \hat{\delta} \Subset \wedge^{n-1} IC_{\lambda_2} * IC_{\lambda_2} * \hat{\delta} \simeq IC_{\lambda_2} * IC_{\lambda_n} * \hat{\delta} \simeq IC_{\lambda_2} * \mathbb{D}\hat{\delta}.$$

and so

$$\wedge^n IC_{\lambda_2} * (\mathbb{D}\hat{\delta})^{-1} \Subset IC_{\lambda_2} * \hat{\delta}.$$

But it is easy to see that the last complex is perverse and indecomposable, so $\wedge^n IC_{\lambda_2} = 0$, and we are done. □

Proof of b). This now follows from a) and d). □

4.3 Lie algebra version of the main theorem

The proof above can be straightforwardly adopted to the case of the sheaves on the Lie algebra $\mathfrak{gl}_n$ of GL_n, as in [Lus87], [Mir04].

Identifying the nilpotent cone $\mathcal{N}$ with the unipotent cone $\mathcal{N}_u$, we get perverse sheaves IC_λ on $\mathfrak{gl}_n$, as in Section 4.1.

Category $D^b(\mathfrak{gl}_n)$ is equipped with convolution with respect to the additive group structure on $\mathfrak{gl}_n$. Since this group structure is abelian, $D^b(\mathfrak{gl}_n)$ is naturally a symmetric braided category. Let $\mathfrak{spr}_P \in D^b(\mathfrak{gl}_n)$ be the perverse sheaf $IC_{\lambda_1} \oplus IC_{\lambda_2}$. We have the following

Theorem 2. *a)* $\wedge^k \mathfrak{spr}_P \simeq (IC_{\lambda_k} \oplus IC_{\lambda_{k+1}})$, *for* $0 < k < n$.

b) $\wedge^n \mathfrak{spr}_P \simeq IC_{\lambda_n}$

c) $\wedge^{n+1} \mathfrak{spr}_P = 0$.

Proof. The proof of part a) is the straightforward adaptation of the proof for the Lie group. So only the part c) needs a separate proof. In the Lie algebra case it is much simpler. Recall the Fourier transform functor

$$\mathrm{FT} : D^b_{\mathbb{G}_m}(\mathfrak{gl}_n) \to D^b_{\mathbb{G}_m}(\mathfrak{gl}_n)$$

from [Lus87], [Mir04]. Here $D^b_{\mathbb{G}_m}(\mathfrak{gl}_n)$ stands for the derived category equivariant with respect to the contracting action of $\mathbb{G}_m$ on $\mathfrak{gl}_n$. FT intertwines the convolution monoidal structure with the regular tensor product.

Let

$$\widetilde{\mathfrak{gl}}_{n,P} = \{(x, l) \in \mathfrak{gl}_n \times \mathbb{P}(V_n) : xl \subset l\}$$

be the parabolic Grothendieck-Springer variety. We have the natural projection

$$\pi : \widetilde{\mathfrak{gl}}_{n,P} \to \mathfrak{gl}_n,$$

and

$$\mathrm{FT}(\mathfrak{spr}_P) = \pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{gl}}_{n,P}}[n^2].$$

But all the fibers of π are unions of projective spaces having the total dimension of cohomology bounded by n. So, with respect to the monoidal structure given by tensor product, we have

$$\wedge^{n+1} \underline{\mathbb{C}}_{\widetilde{\mathfrak{gl}}_{n,P}}[n^2],$$

and so

$$\wedge^{n+1} \mathfrak{spr}_P = 0$$

with respect to the additive convolution. □

4.4 Another description of the isomorphism in the main theorem

We will need the following description of the isomorphism

$$\wedge^k Spr_P \simeq IC_{\lambda_k} \oplus IC_{\lambda_{k+1}}.$$

Let L be a Levi subgroup of P. We have the natural morphism of functors

$$Spr_P * (\cdot) = \chi_P \mathfrak{hc}_P(\cdot) \to \chi_P \delta_{P*} \delta_P^* \mathfrak{hc}_P(\cdot) = \mathrm{Ind}_L^G \mathrm{Res}_L^G(\cdot),$$

and so a morphism

$$\phi_k : \wedge^k Spr_P \to \left(\mathrm{Ind}_L^G \mathrm{Res}_L^G\right)^k (IC_{\lambda_1}) \tag{4.3}$$

Lemma 4.4.1. *ϕ_k gives an isomorphism to the corresponding summand of the right hand side of* (4.3).

Proof. For $k = 1$ the statement is obvious. Assume that it is known for $k = k_0$. We have a morphism

$$Spr_P * \wedge^{k_0} Spr_P \to \mathrm{Ind}_L^G \mathrm{Res}_L^G(\wedge^{k_0} Spr_P).$$

We want to show that, restricted to the corresponding direct summands, it gives an isomorphism. But the morphism above gives an isomorphism of the stalks at the unit $e \in G$, and the summands in question are semi-simple perverse sheaves with all irreducible summands having non-zero stalks at e, so we are done. □

Chapter 5

Extension to the functor from the perfect derived category

Since we have $G^\vee = G$, in this section we will omit the Langlands duality from all notations.

Define Jucys-Murphy sheaves $\mathbb{L}_i$ to be

$$\mathbb{L}_1 = \Delta_e^{\hat{\mathcal{L}}_1}, \mathbb{L}_k = \Delta_{s_{k-1}}^{\hat{\mathcal{L}}_1} * \mathbb{L}_{k-1} * \Delta_{s_{k-1}}^{\hat{\mathcal{L}}_1},$$

where $s_i = (i\ i+1)$, cf. Chapter 2.1. For a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ of GL_n write

$$\mathbb{L}_\lambda = \mathbb{L}_2^{*\lambda_2} * \dots * \mathbb{L}_n^{*\lambda_n}.$$

Note that we have $\mathbb{L}_\lambda * \mathbb{L}_\mu \simeq \mathbb{L}_{\lambda+\mu}$, since the fact that Jucys-Murphy elements commute follows just from the braid relations.

In this section we will prove

Theorem 3. *There is a monoidal functor $\varpi : D^G_{perf}(\hat{St}) \to \hat{D}_f$ satisfying the following:*

a) $\varpi(\mathcal{O}_{\hat{St}}) = \hat{\mathcal{T}}_{w_0}$.

b) $\varpi(\mathcal{O}_{\hat{St}}(\lambda, \mu)) = \mathbb{L}_\lambda * \hat{\mathcal{T}}_{w_0} * \mathbb{L}_\mu$ *for any two weights λ, μ of GL_n.*

c) $\varpi(V \otimes \mathcal{O}_{St}) = (\mathfrak{hc} \circ \varpi_Z(V)) * \hat{\mathcal{T}}_{w_0}$ *for any representation V of G.*

Remark. This theorem can be thought of as a categorification of the projection Π_{perf}, see Section 2.3.

Proof. We employ the general construction of [Bez16]. Let $\mathfrak{t}$ be the Lie algebra of T.

Let $C_{\tilde{\mathfrak{g}}}$ be the preimage in $\mathfrak{g} \times Y$ of $\tilde{\mathfrak{g}}$ under the natural projection $\mathfrak{g} \times Y \to \mathfrak{g} \times \mathcal{B}$. Since the right T-action on Y is free, we have $Coh^{G \times T}(C_{\tilde{\mathfrak{g}}}) = Coh^G(\tilde{\mathfrak{g}})$.

Let $\overline{Y} = \operatorname{Spec} \Gamma(\mathcal{O}_Y)$ be the affine closure of Y.

Now pick a representation V of G, such that

a) V is a multiplicity-free sum of irreducible representations;

b) Y is a locally closed subvariety of V, namely an orbit of a highest-weight vector;

c) $\overline{Y}$ is a closure of Y inside V.

Choose an action of $\mathfrak{t}$ on V so that $t \in \mathfrak{t}$ acts on an irreducible summand with the highest weight λ by a constant $\langle \lambda, t \rangle$.

Let $\overline{C}_{\tilde{\mathfrak{g}}}$ be a closed subscheme of $\mathfrak{g} \times \mathfrak{t} \times \overline{Y}$ given by

$$\overline{C}_{\tilde{\mathfrak{g}}} = \{(x, t, v) \in \mathfrak{g} \times \mathfrak{t} \times \overline{Y} \subset \mathfrak{g} \times \mathfrak{t} \times V : x(v) = t(v)\}.$$

$C_{\tilde{\mathfrak{g}}}$ becomes an open subset in $\overline{C}_{\tilde{\mathfrak{g}}}$, given by (x, t, v) with v having a non-zero projection to all irreducible summands of V.

Let $\overline{C}_{\mathrm{St}}$ be the preimage of the diagonal under the map $\overline{C}_{\tilde{\mathfrak{g}}} \times \overline{C}_{\tilde{\mathfrak{g}}} \to \mathfrak{g} \times \mathfrak{g}$. We have an open subset $C_{\mathrm{St}} \subset \overline{C}_{\mathrm{St}}$ with a free T^2-action, and $C_{\mathrm{St}}/T^2 = St$

By an action of a monoidal category $\mathcal{A}$ on a triangulated category $\mathcal{B}$ we mean a monoidal functor $\alpha : \mathcal{A} \to End(\mathcal{B})$, with all $\alpha(x)$ respecting the triangulated structure.

For a variety X with an action of an algebraic group H, let $Coh^H_{fr}(X)$ be the full subcategory of $Coh^H(X)$, consisting of objects of the form $V \otimes \mathcal{O}_X$, $V \in \mathrm{Rep}(H)$.

We have the following

Proposition ([Bez16], Corollary 18 and Chapter 4.4.1)**.** *Let $\mathcal{C}$ be a $\mathbb{C}$-linear additive monoidal category. Suppose we are given*

a) A tensor functor $F : \mathrm{Rep}(G \times T) \to End(\mathcal{C})$, respecting the triangulated structure.

b) A tensor endomorphism E of $F|_{\mathrm{Rep}(G)}$, satisfying

$$E_{V_1 \otimes V_2} = E_{V_1} \otimes Id_{F(V_2)} + Id_{F(V_1)} \otimes E_{V_2}.$$

c) An action of $\mathcal{O}(\mathfrak{t})$ on F by endomorpisms, so that for $f \in \mathcal{O}(\mathfrak{t})$ we have $f_{V_1 \otimes V_2} = f_{V_1} \otimes Id_{V_2} = Id_{V_1} \otimes f_{V_2}$.

d) A "lowest weight arrow" $w_\lambda : F(V_{w_0\lambda}) \to F(\lambda)$ making the following diagrams commutative:

$$\begin{array}{ccc} F(V_{w_0\lambda} \otimes V_{w_0\mu}) & \longrightarrow & F\left(V_{w_0(\lambda+\mu)}\right) \\ {\scriptstyle w_\lambda \otimes_{\mathcal{C}} w_\mu}\downarrow & & \downarrow{\scriptstyle w_{\lambda+\mu}} \\ F(\lambda) \otimes_{\mathcal{C}} F(\mu) & \xrightarrow{\sim} & F(\lambda + \mu) \end{array}$$

$$\begin{array}{ccc} F(V_{w_0\lambda}) & \xrightarrow{w_\lambda} & F(\lambda) \\ {\scriptstyle E_\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ F(V_{w_0\lambda}) & \xrightarrow{w_\lambda} & F(\lambda) \end{array}$$

where the right vertical map is the action of the element $\lambda \in \mathfrak{t} \subset \mathcal{O}(\mathfrak{t})$ coming from c).

Then the tensor functor F extends uniquely to an action of $Coh_{fr}^{G\times T}(\overline{C}_{\tilde{\mathfrak{g}}})$ on $\mathcal{C}$. If we have two copies of the above data, we get an action of $Coh_{fr}^{G^2\times T^2}(\overline{C}_{\tilde{\mathfrak{g}}} \times \overline{C}_{\tilde{\mathfrak{g}}})$. In this case, assume, moreover, that

e) the action of $\mathrm{Rep}(G \times G)$ factors through the diagonal $G \to G \times G$.

f) this action of $\mathrm{Rep}(G)$ is by product with central objects in a braided category $\mathcal{C}$, with endomorphism E from b) respecting the central structure.

Then we have an action of $Coh_{fr}^{G\times T^2}(\overline{C}_{St})$, extending the above data.

We now indicate the structures listed in the above proposition, constructing an action of $Coh_{fr}^{G\times T^2}(\overline{C}_{\mathrm{St}})$ on $\hat{D}_f$.

First, the action of $\mathrm{Rep}(G \times G)$ is given by restriction to the diagonal and convolution with central sheaves from Corollary 4.2.1. Action of $\mathrm{Rep}(T \times T)$ is given by left and right convolutions with Jucys-Murphy sheaves $\mathbb{L}_\lambda$.

Action of $(f \otimes g) \in \mathcal{O}(\mathfrak{t} \times \mathfrak{t})$ is given by the action of f and g coming from left and right monodromy, respectively.

We construct the rest of the data and prove the needed compatibilities in the following sections.

5.1 Lowest weight arrow and monodromy endomorphism

For $k < n$ consider the standard representation V_k of $\mathrm{GL}_k = \mathrm{GL}(V_k)$. Let $\mathrm{triv}_{\mathrm{GL}_{n-k}}$ stand for the trivial representation of GL_{n-k}. Consider

$$\boldsymbol{\varsigma}_k = \varpi_Z(\wedge^k V_k) \boxtimes \varpi_Z(\mathrm{triv}_{\mathrm{GL}_{n-k}}),$$

a (pro-)character sheaf on the Levi subgroup $L_k = \mathrm{GL}_k \times \mathrm{GL}_{n-k}$ of $G = \mathrm{GL}_n$. Here the ligature $\boldsymbol{\varsigma}$ stands for "*St*einberg". Note that we abuse notation here, denoting by ϖ_Z functors $G = \mathrm{GL}_l$ with different l. It should always be clear what we mean.

We have $\varpi_Z(\wedge^k V_n) = \mathrm{Ind}_L^G(\boldsymbol{\varsigma}_k)$.

$\boldsymbol{\varsigma}_k$ is the IC-extension of the local system on the set of regular semisimple elements $\mathrm{GL}_k^{reg} \times \mathrm{GL}_{n-k}^{reg}$, given by a module over the extended affine Weyl group $(S_k \times S_{n-k}) \ltimes \mathbb{Z}^n$, with a pro-unipotent action of $\mathbb{Z}^n$. Or, passing to the logarithms of the monodromy, by a module over

$$A_k = \mathbb{C}[S_k \times S_{n-k}] \ltimes S(\mathfrak{t}) = \mathbb{C}[S_k \times S_{n-k}] \ltimes \mathbb{C}[e_1, \dots, e_n].$$

Note that the action of $(e_1 + \dots + e_k) \in A_k$ is by a module endomorphism. Let P_k be a parabolic subgroup corresponding to L_k. We have

$$\mathfrak{hc} \circ \varpi_Z(\wedge^k V_n) = \mathfrak{hc}_B^{P_k} \circ \mathfrak{hc}_{P_k} \circ \chi_{P_k} i_{L_k}(\boldsymbol{\varsigma}_k).$$

Let ω_k be the lowest weight of $\wedge^k V_n$. We have

$$\mathfrak{hc}_B^{P_k} \circ i_{L_k}(\boldsymbol{\varsigma}_k) \simeq \mathbb{L}_{\omega_k} = \mathbb{L}_1 * \cdots * \mathbb{L}_k,$$

see the proof of Theorem 1. We also have the natural morphism

$$r_k : \mathfrak{hc}_{P_k} \circ \chi_{P_k} i_{L_k}(\boldsymbol{\varsigma}_k) \to i_{L_k}(\boldsymbol{\varsigma}_k),$$

see Section 3.3.

Combining, we get an endomorphism

$$\mathrm{Ind}_{L_k}^G(e_1 + \cdots + e_k) =: \varepsilon_k \in End(\varpi_Z(\wedge^k V_n)),$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \varpi(\wedge^k V_n) & \xrightarrow{w_{\omega_k}} & \mathbb{L}_{\omega_k} \\ {\scriptstyle E_k}\downarrow & & \downarrow{\scriptstyle \omega_k} \\ \varpi(\wedge^k V_n) & \xrightarrow{w_{\omega_k}} & \mathbb{L}_{\omega_k} \end{array}$$

Here $w_{\omega_k} = \mathfrak{hc}_B^{P_k}(r_k), E_k = \mathfrak{hc}(\varepsilon_k)$, and ω_k comes from a monodromy action on $\mathbb{L}_{\omega_k}$.

We define w_{ω_k} to be the lowest weight arrows for the images of fundamental representations $\wedge^k V_n$. Note that, since the lowest weight vector of any representation can be given by a product of the lowest weight vectors of fundamental representations in their tensor product, we may define, for a lowest weight $\lambda = \sum a_i \omega_i, w_\lambda = \prod_i w_{\omega_i}^{*a_i}$.

Now the condition b) of Proposition 5 recovers E_V from E_1 in a unique way. The only thing left to check is

Lemma 5.1.1. *E_{ω_k} obtain from E_1 by imposing condition b) is equal to E_k defined above.*

Proof. Recall that we have a natural morphism of functors

$$Spr_P * (\cdot) \to \mathrm{Ind}_P^G \mathrm{Res}_P^G(\cdot),$$

which gives a morphism

$$Spr_P^{*k}(\cdot) \to \left(\mathrm{Ind}_P^G \mathrm{Res}_P^G\right)^k (\cdot).$$

By the result of Section 4.4, it gives an isomorphism on the direct summands corresponding to the exterior power. If a character sheaf $\mathcal{F}$ is given by the IC-extension of a local system corresponding to a $\mathbb{C}[S_n] \ltimes \mathbb{Z}^n$-module M, then $\mathrm{Ind}_P^G \mathrm{Res}_P^G \mathcal{F}$ is given by the IC-extension corresponding to a $\mathbb{C}[S_n] \ltimes \mathbb{Z}^n$-module $\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} M$.

Let V_n be the n-dimensional permutation representation of S_n. Choosing an isomorphism of S_n-modules

$$\mathbb{C}[S_n]^{\otimes_{\mathbb{C}[S_{n-1}]} k} \otimes_{\mathbb{C}[S_{n-1}]} \mathrm{triv}_{S_n} \simeq V_n^{\otimes k}$$

and comparing the actions under this identification, we get that the action of E_{ω_k} on the corresponding direct summand is the same as given by E_k. $\square$

5.2 Filtration by Jucys-Murphy sheaves

We have that $P_k \supset B$ is a parabolic subgroup corresponding to the subset $J = \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_{n-1}\} \subset I$. By Lemmas 3.1.1, 3.2.2 and the discussion in Section 2.2, we get

Lemma 5.2.1. $(\mathfrak{hc} \circ \varpi_Z)(\wedge^k V_n) \in \langle \mathbb{L}_{\lambda'} \rangle_{\lambda' \in W\omega_k}$.

5.3 Passage from $\overline{C}_{\mathbf{St}}$ to St

We constructed the action of $Coh_{fr}^{G\times T^2}\overline{C}_{\mathrm{St}}$ on $\hat{D}_f$. Consider the action on the object $\hat{\mathcal{T}}_{w_0}$. This gives a functor

$$\overline{\varpi} : Coh_{fr}^{G\times T^2}\overline{C}_{\mathrm{St}} \to \hat{D}_f.$$

Since $\mathrm{Rep}(G \times T^2)$ act by convolution with objects having a filtration by products of $\Delta_w^{\hat{\zeta}_1}$, discussion of $\hat{\mathcal{T}}_{w_0}$ in Section 3.4 gives that $\overline{\varpi}$ in fact sends $Coh_{fr}^{G\times T^2}\overline{C}_{\mathrm{St}}$ to $\hat{\mathcal{T}}$. Thus, we get a functor

$$Ho(Coh_{fr}^{G\times T^2}\overline{C}_{\mathrm{St}}) \to Ho(\hat{\mathcal{T}}) \simeq \hat{D}_f.$$

Finally, we employ the following proposition from [AB09]. In any tensor category over a field of characteristic zero one can construct a Koszul complex $K_{\phi,d}$ associated to any morphism $\phi : V \to L$ and $d \in \mathbb{Z}^{>0}$, namely the complex

$$0 \to \wedge^d(V) \to \wedge^{d-1}(V) \otimes L \to \cdots \wedge^{d-i}(V) \otimes Sym^i(L) \to \cdots \to Sym^d(L) \to 0.$$

Let K_λ be the Koszul complex $K_{w_\lambda, \dim V_\lambda}$ in $\left(Coh_{fr}^{G\times T^2}\overline{C}_{\mathrm{St}}, \otimes_{\mathcal{O}_{\overline{C}_{\mathrm{St}}}}\right)$

Proposition 1 ([AB09], Lemma 20)**.** *Assume there is an action of*

$$a : Ho(Coh_{fr}^{G\times T^2}\overline{C}_{St}) \curvearrowright \mathcal{C}$$

such that $a(K_\lambda)x = 0$ for all λ and some object $x \in \mathrm{Ob}(\mathcal{C})$. Then the functor

$$\bar{F} : Ho(Coh_{fr}^{G\times T^2}\overline{C}_{St}) \to \mathcal{C}, y \mapsto a(y)x$$

factors through the functor

$$F : D_{perf}^G(St) \to \mathcal{C}.$$

Functor $\overline{\pi}$ satisfies the condition of the above proposition, because, on an object of the form $V \otimes \mathcal{O}, V \in \mathrm{Rep}(G \times T^2)$, it is given by $V \otimes \hat{\mathcal{T}}_{w_0}$. We have

$$\phi(K_\lambda) = k_\lambda \otimes \mathcal{T}_{w_0} = 0,$$

where k_λ is the (obviously acyclic) Koszul complex in $\mathrm{Rep}(G \times T^2)$.

Now, as in [Bez16], the fact that the action of E is pro-nilpotent implies that we get a functor

$$\varpi : D^G_{\mathrm{perf}}(\hat{\mathrm{St}}) \to \hat{D}_f.$$

5.4 Compatibility with the monoidal structure

It is shown in [Bez16] that the convolution $\mathcal{F} * \mathcal{G}$ in $D^G_{\mathrm{perf}}(\hat{\mathrm{St}})$ can be computed by taking total complex of a bicomplex $\{\mathcal{F}^i * \mathcal{G}^j\}$, where $\{\mathcal{F}^i\}, \{\mathcal{G}^j\}$ are complexes representing $\mathcal{F}, \mathcal{G}$ in $Ho(Coh_{fr}\hat{\mathrm{St}})$. Similarly, the convolution $\mathcal{F} * \mathcal{G}$ in $\hat{D}_f$ can be computed by taking total complex of a bicomplex $\{\mathcal{F}^i * \mathcal{G}^j\}$, where $\{\mathcal{F}^i\}, \{\mathcal{G}^j\}$ are complexes representing $\mathcal{F}, \mathcal{G}$ in $Ho(\hat{\mathcal{T}})$. Thus, we only need to check

$$\varpi(\mathcal{O}_{\hat{\mathrm{St}}} * \mathcal{O}_{\hat{\mathrm{St}}}) \simeq \hat{\mathcal{T}}_{w_0} * \hat{\mathcal{T}}_{w_0},$$

which is again a result of op.cit. □

Appendix A

Remark on the Kirillov-Pak character formula

In this Appendix we discuss a formula from [KP90]. We include it for the sake of completeness, because it was a part of our initial motivation for choosing the parabolic Springer sheaf as the image of the standard representation of GL_n in Theorem 1. We guessed it independently using CHEVIE package [GHL$^+$96] for GAP3 computer algebra system [S$^+$97].

Let V be the standard n-dimensional permutation representation of S_n, and consider

$$R = \mathbb{C}[x_1, \dots, x_n]$$

as a graded S_n-module, with S_n-action permuting the variables.

Note that, on the one hand, local system defining the character sheaf $Spr_P * \hat{\delta}$ is given by $R \otimes V$ considered as a $\mathbb{C}[S_n] \ltimes R$-module. On the other hand, $\mathfrak{hc}(Spr_P * \hat{\delta})$ caregorifies the sum of Jucys-Murphy elements, and this is known to act in the irreducible module E_λ of $\mathbb{H}_n$ corresponding to a Young diagram λ by the constant

$$\sum_{\square_i \in \lambda} q^{c(\square_i)} = \mathbf{e}_1 \left(q^{c(\square_1)}, \dots, q^{c(\square_n)} \right),$$

where the sum is taken over the boxes of diagram λ, $c(\cdot)$ stands for the content of a box, and $q = v^2$, see [IO05].

We can write the same expressions for $\wedge^k Spr_P * \hat{\delta}$ involving elementary symmetric polynomials $\mathbf{e}_k$ in Jucys-Murphy sheaves, see 2.2.

Let Λ be the ring of symmetric functions, and let χ_q stand for the Frobenius character from representations of S_n to $\Lambda[[q]]$. Let s_λ be the Schur symmetric function corresponding to a partition λ. Define power series $Q_\lambda \in \mathbb{Z}[[q]]$ by

$$\chi_q(R) = \sum Q_\lambda s_\lambda,$$

The above discussion suggests the following:

Proposition ([KP90])**.**

$$\chi_q(\wedge^k V \otimes R) = \sum_\lambda \mathbf{e}_k\left(q^{c(\square_1(\lambda))}, \dots, q^{c(\square_n(\lambda))}\right) Q_\lambda s_\lambda.$$

Proof. Indeed, it is well-known that

$$Q_\lambda = s_\lambda(1, q, q^2, \dots) = q^{b(\lambda)} \frac{1}{\prod_{\square_i \in \lambda}(1 - q^{h(\square_i)})},$$

where $b(\lambda) = \sum_i (i-1)\lambda_i$, and $h(\cdot)$ stands for the hook length, so the proposition is precisely the statement of [KP90] in the coefficient of s^k. □

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